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Simple shear flow round a rigid sphere: inertial effects and suspension rheology

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An analysis is presented of the flow field near a neutrally-buoyant rigid spherical particle immersed in an incompressible Newtonian fluid which, at large distances from the particle, is undergoing simple shear flow. Subject to conditions of continuity of stress at the particle surface and to conditions of zero net torque and zero net force on the sphere, the effect of fluid inertia on the velocity and pressure fields in the vicinity of the particle has been computed to $O(R^{\frac{3}{2}})$, where $R = a^2G/\nu$ is a shear Reynolds number, a being the sphere radius, G the velocity gradient in the free stream (taken to be a positive number), and ν the kinematic viscosity.

Some streamlines have been computed and plotted. These illustrate how the fore–aft symmetry of the creeping-motion solution is destroyed when one includes inertial effects.

Knowledge of the velocity and pressure fields enables one to compute the effect of inertial forces in suspension rheology. The results include a correction to the Einstein viscosity law to $O(R^{\frac{3}{2}})$ for a dilute (non-interacting) suspension of spheres. In addition it is found that inertial effects give rise to a non-isotropic normal stress.

1. Introduction

Development and application of singular perturbation techniques by Kaplun & Lagerstrom (1957) and Proudman & Pearson (1957) to the classical Stokes problem have resulted in renewed and continuing interest in the effects of inertia on low Reynolds number flows of particulate systems. Of special interest is the application of singular perturbation methods to problems dealing with the behaviour of particulate matter in shear flows. Saffman (1965) has shown that a sphere, when moving relative to a fluid undergoing uniform simple shear, experiences a lift force transverse to the direction of fluid and particle motion. The lift is a direct consequence of inertial effects. Harper & Chang (1968) have generalized Saffman's analysis to the case of any three-dimensional body and

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have related a lift tensor to the Stokes translation dyadic for the body. An essential feature of the papers of Saffman and of Harper & Chang is the role played by the velocity of the particle relative to the undisturbed fluid velocity at the position occupied by the particle. Indeed, the lift force is proportional to this relative velocity.

In the present paper we treat a related but different problem. Consider a neutrally buoyant spherical particle immersed in an incompressible Newtonian fluid which, at large distances from the particle, is in a state of simple shear flow. Subject to conditions of continuity of stress and velocity at the particle surface and to conditions of zero net torque and zero net force on the sphere, we ask the following question: What is the effect of fluid inertia on the velocity and pressure fields in the vicinity of the particle? This question is asked for two reasons. First, it is of interest to know the extent to which an accounting of inertia alters the well-known creeping-motion solution, as presented, for example, by Landau & Lifshitz (1959, p. 76). Secondly, one can apply knowledge of the velocity field, including inertial effects, to computation of the rheological behaviour of a dilute suspension of spheres flowing at small but non-zero Reynolds number.

We have computed the fluid velocity field near the particle to $O(R^{\frac{3}{2}})$, where $R = a^2G\rho/\mu$ is a shear Reynolds number based upon particle radius a, velocity gradient of unperturbed fluid G, fluid density ρ and fluid viscosity μ. Essential features of the outer solution were found by application of a Fourier transformation procedure similar to that employed by Saffman, who in turn modified a device introduced by Childress (1964). The flow field near the particle was then used to compute the constitutive equation, including terms to $O(R^{\frac{3}{2}})$, for a system of dilute (i.e. non-interacting) neutrally-buoyant rigid spheres uniformly distributed in an incompressible Newtonian fluid. Not only is the shear viscosity altered from the classical result of Einstein, but the system exhibits normal stresses which are caused by inertial effects. The results for a fluid with viscosity μ are, to $O(\phi)$,

$$\mu_s = \mu[1+\phi(\tfrac{5}{2}+1\cdot34R^{\frac{3}{2}})],$$
$$t'_{xx}-t'_{zz} = \mu G\phi R[-\tfrac{4}{3}+0\cdot035R^{\frac{1}{2}}],$$
$$t'_{yy}-t'_{zz} = \mu G\phi R[\tfrac{2}{3}-0\cdot252R^{\frac{1}{2}}].$$

μ_s is the suspension viscosity and t'_{ij} refers to components of the stress tensor with respect to a co-ordinate system in which the bulk flow has the velocity $(v_x, v_y, v_z) = (Gy, 0, 0)$. Volume fraction of solids is ϕ, and $G \geqslant 0$ is the (constant) average shear rate of the suspension.

The problem is formulated in the next section. This is followed in § 3 by development of expansions for the velocity and pressure fields and an exposition of the matching technique. Some sample streamlines around a single sphere are presented in § 4, while the subject of suspension rheology is considered in § 5.

2. Formulation of the problem

An incompressible Newtonian fluid is in steady shear flow past a neutrally-buoyant rigid sphere which is freely suspended in the fluid. We describe the flow field with respect to a fixed co-ordinate system the origin of which is at the centre of the sphere. Fluid velocity at infinity is taken to be

$$\mathbf{u}'_\infty = Gy'\mathbf{e}_x,$$

where $\mathbf{e}_x$ is a unit vector in the x' direction. Subject to the condition of free suspension of the sphere, i.e. no net force or torque on the sphere, we allow for rotation $\mathbf{\Omega}'$ about the centre of the sphere. One can also include a velocity of translation of the sphere $\mathbf{V}'$. However, because of symmetry we can set $\mathbf{V} = \mathbf{0}$ in the present problem. Governing equations for the fluid velocity $\mathbf{u}'$ are the steady-state Navier–Stokes equation and the continuity equation together with the boundary conditions of no slip at the particle surface and the requirements that $\mathbf{u}' \to \mathbf{u}'_\infty$ and $p' \to p'_\infty =$ constant as $|\mathbf{r}'| \to \infty$. It is convenient to define the following non-dimensional quantities:

$$\mathbf{r} = \mathbf{r}'/a, \quad \mathbf{u} = \mathbf{u}'/aG, \quad p = p'/\mu G, \quad p_\infty = p'_\infty/\mu G,$$
$$\mathbf{V} = \mathbf{V}'/aG = \mathbf{0} \quad \text{and} \quad \mathbf{\Omega} = \mathbf{\Omega}'/G.$$

In terms of these dimensionless variables the governing equations are

$$-\nabla p + \nabla^2\mathbf{u} = R\mathbf{u}\,.\,\nabla\mathbf{u} \tag{2.1 a}$$

$$\nabla\,.\,\mathbf{u} = 0, \tag{2.1 b}$$

with boundary conditions

$$\mathbf{u} = \mathbf{\Omega}\times\mathbf{r} \quad \text{at} \quad |\mathbf{r}| = 1, \tag{2.2 a}$$

$$\mathbf{u} \to y\mathbf{e}_x \quad \text{and} \quad p \to p_\infty \quad \text{as} \quad |\mathbf{r}| \to \infty, \tag{2.2 b}$$

$\mathbf{\Omega}$ being determined by the condition of free suspension.

3. Inner and outer expansions

(a) Inner expansion

Following earlier workers we assume that in the inner region of the flow, i.e. where $r = |\mathbf{r}| = O(1)$, the expansions for the flow variables $\mathbf{u}$, p, $\mathbf{V}$ and $\mathbf{\Omega}$ are of the form

$$\{\mathbf{u}, p, \mathbf{V}, \mathbf{\Omega}\} = \sum_{n=0}^{\infty} f_n(R)\{\mathbf{u}_n, p_n, \mathbf{V}_n, \mathbf{\Omega}_n\}, \tag{3.1 a}$$

with

$$\lim_{R\to 0}\,[f_{n+1}(R)/f_n(R)] = 0. \tag{3.1 b}$$

In accord with custom we designate (3.1 a) as the inner or Stokes expansion and take $f_0(R) = 1$. Individual terms of the expansion (3.1 a) are required to satisfy (2.1) and the no-slip condition (2.2 a). Since (3.1) is invalid at large values of r, we replace the boundary condition (2.2 b) by the requirement that (3.1) match an expansion which is valid far from the sphere.

Substituting (3.1 a) into (2.1) and (2.2 a) and equating coefficients of the same order of magnitude in R, one obtains the governing equations for each pair $\{\mathbf{u}_n, p_n\}$:

$$-\nabla p_n + \nabla^2\mathbf{u}_n = \sum_{\substack{l=0 \\ (Rf_l f_m = f_n)}} \sum_{m=0} (\mathbf{u}_l\,.\,\nabla\mathbf{u}_m), \tag{3.2 a}$$

$$\nabla\,.\,\mathbf{u}_n = 0. \tag{3.2 b}$$

Since $\mathbf{V}_n = \mathbf{0}$, the surface boundary condition is

$$\mathbf{u}_n = \mathbf{\Omega}_n \times \mathbf{r} \quad \text{at} \quad r = 1. \tag{3.3}$$

The double summation in (3.2 a) is restricted to those terms for which

$$Rf_l(R) f_m(R) = f_n(R).$$

The condition (3.1 b) ensures that l and m must be less than n. Consequently, for any value of n, the summation term only contains contributions from lower-order solutions, and in an iteration scheme these contributions will be known. Thus (3.2a) constitutes a set of linear inhomogeneous equations, each of which is formally equivalent to the Stokes equation modified for the presence of an external volume force (see, for example, Brenner 1966). Let

$$\mathbf{g}_n\,(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-1}) = -\sum_l \sum_m (\mathbf{u}_l \,.\, \nabla \mathbf{u}_m)$$

denote the equivalent dimensionless force per unit volume which is exerted on the fluid at some point by the surroundings. Then

$$-\nabla p_n + \nabla^2 \mathbf{u}_n = -\mathbf{g}_n. \tag{3.2a'}$$

Because of the linearity of (3.2) and (3.3) the solutions can be decomposed into homogeneous and particular parts,

$$\mathbf{u}_n = \mathbf{u}_{nh} + \mathbf{u}_{np}, \quad p_n = p_{nh} + p_{np}, \tag{3.4a, b}$$

where

$$-\nabla p_{nh} + \nabla^2 \mathbf{u}_{nh} = \mathbf{0}, \quad \nabla \,.\, \mathbf{u}_{nh} = 0, \tag{3.5a, b}$$

with the boundary condition $\mathbf{u}_{nh} = \mathbf{\Omega}_n \times \mathbf{r}$ at $r = 1$, and

$$-\nabla p_{np} + \nabla^2 \mathbf{u}_{np} = -\mathbf{g}_n, \quad \nabla \,.\, \mathbf{u}_{np} = 0, \tag{3.6a, b}$$

with the boundary condition $\mathbf{u}_{np} = \mathbf{0}$ at $r = 1$. The solutions $\{\mathbf{u}_{nh}, p_{nh}\}$ and $\{\mathbf{u}_{np}, p_{np}\}$ are given in appendix A.

The condition of free suspension is similarly decomposed. The hydrodynamic force $\mathbf{F}$ and torque $\mathbf{T}$ (about the centre of the sphere) are expanded as follows:

$$\{\mathbf{F}, \mathbf{T}\} = \sum_{n=0}^{\infty} f_n(R) \{\mathbf{F}_{nh} + \mathbf{F}_{np}, \mathbf{T}_{nh} + \mathbf{T}_{np}\}. \tag{3.7}$$

We require

$$\mathbf{F}_{nh} + \mathbf{F}_{np} = \mathbf{T}_{nh} + \mathbf{T}_{np} = \mathbf{0}, \tag{3.8}$$

where the subscripts nh and np designate contributions arising from the velocity and pressure fields satisfying (3.5) and (3.6), respectively. The requirement (3.8) fixes angular velocity $\mathbf{\Omega}_n$ in the boundary condition to (3.5).

(*b*) *Outer expansion*

The flow field far from the particle is conveniently described by changing variables so that the Reynolds number does not appear explicitly in the equation of motion. Thus we scale independent and dependent variables by the transformations

$$\tilde{\mathbf{r}} = R^{\frac{1}{2}}\mathbf{r}, \quad \tilde{\mathbf{u}} = R^{\frac{1}{2}}\mathbf{u} \quad \text{and} \quad \tilde{p} = p, \tag{3.9}$$

so that (2.1) becomes

$$-\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2 \tilde{\mathbf{u}} = \tilde{\mathbf{u}} \,.\, \tilde{\nabla}\tilde{\mathbf{u}}, \quad \tilde{\nabla} \,.\, \tilde{\mathbf{u}} = 0. \tag{3.10a, b}$$

The outer expansion (Oseen expansion) is assumed to have the form

$$\{\tilde{\mathbf{u}}, \tilde{p}\} = \sum_{n=0}^{\infty} F_n(R)\{\tilde{\mathbf{u}}_n, \tilde{p}_n\}, \tag{3.11 a}$$

where

$$\lim_{R\to 0}\,[F_{n+1}/F_n] = 0 \tag{3.11 b}$$

and we take $F_0(R) = 1$.

Terms of the outer expansion (3.11) are required to satisfy the differential equations (3.10) and, as $\tilde{r} \equiv |\tilde{\mathbf{r}}| \to \infty$, must approach a uniform shear field. However, the no-slip condition on the sphere is replaced by the condition that (3.11) must match with the Stokes expansion (3.1).

Upon substitution of (3.11 a) into (3. 10) and (2.2 b) one obtains, after equating coefficients of the same order of magnitude of R in each equation, the governing equations for each pair $\{\tilde{\mathbf{u}}_n, \tilde{p}_n\}$ in the outer expansion; viz.

$$-\tilde{\nabla}\tilde{p}_n + \tilde{\nabla}^2\tilde{\mathbf{u}}_n - \tilde{y}\frac{\partial\tilde{\mathbf{u}}_n}{\partial\tilde{x}} - \tilde{u}_{ny}\,\mathbf{e}_x = \sum_{\substack{l=1\\(F_n=F_lF_m)}}\sum_{m=1}\tilde{\mathbf{u}}_l.\tilde{\nabla}\tilde{\mathbf{u}}_m \tag{3.12 a}$$

$$\tilde{\nabla}.\tilde{\mathbf{u}}_n = 0, \tag{3.12 b}$$

with boundary conditions $$\tilde{\mathbf{u}}_n = \begin{cases}\tilde{y}\mathbf{e}_x & \text{for} \quad n = 0,\\ \mathbf{0} & \text{for} \quad n > 0,\end{cases} \tag{3.13 a}$$

$$\tilde{p}_n = \left.\begin{cases}p_\infty & \text{for} \quad n = 0\\ 0 & \text{for} \quad n > 0\end{cases}\right\} \quad \text{at} \quad \tilde{r}\to\infty. \tag{3.13 b}$$

Following the notation of (3.2 a), we note that the double summation in (3.12 a) is limited to those terms for which $F_l(R)\,F_m(R) = F_n(R)$. Thus l and m must be less than n, and the terms on the right-hand side of (3.12 a) will be known in any iteration scheme. Hence (3.12 a) is a linear inhomogeneous equation of the Oseen type and the right-hand side can be replaced by the force term

$$\mathbf{G}_n(\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_{n-1}) = -\sum_l\sum_m \tilde{\mathbf{u}}_l.\tilde{\nabla}\tilde{\mathbf{u}}_m.$$

For the case $n = 1$ we have $\mathbf{G}_1 = \mathbf{0}$, so that

$$-\tilde{\nabla}\tilde{p}_1 + \tilde{\nabla}^2\tilde{\mathbf{u}}_1 - \tilde{y}\frac{\partial\tilde{\mathbf{u}}_1}{\partial\tilde{x}} - \tilde{u}_{1y}\,\mathbf{e}_x = \mathbf{0}, \tag{3.14 a}$$

$$\tilde{\nabla}.\tilde{\mathbf{u}}_1 = 0, \tag{3.14 b}$$

with boundary conditions (anticipating that $\tilde{\mathbf{u}}_0$ will match the free-stream boundary condition)

$$\tilde{\mathbf{u}}_1 = \mathbf{0} \quad \text{and} \quad \tilde{p}_1 = 0 \quad \text{as} \quad \tilde{r}\to\infty. \tag{3.15}$$

The matching principle can be stated by (Van Dyke 1964, p. 89)

$$R^{\frac{1}{2}}\lim_{r\to\infty}\mathbf{u}(\mathbf{r}) = \lim_{\tilde{r}\to 0}\tilde{\mathbf{u}}(\tilde{\mathbf{r}}) \quad \text{and} \quad \lim_{r\to\infty}p(r) = \lim_{\tilde{r}\to 0}\tilde{p}(\tilde{\mathbf{r}}). \tag{3.16}$$

Now let $\tilde{\mathbf{U}}_1(\tilde{\mathbf{r}})$ and $\tilde{P}_1(\tilde{\mathbf{r}})$ be the contributions of the inner expansion, expressed in outer variable $\tilde{\mathbf{r}}$, to the outer terms $\tilde{\mathbf{u}}_1(\tilde{\mathbf{r}})$ and $\tilde{p}_1(\tilde{\mathbf{r}})$, respectively. Then (3.16) implies

$$\tilde{\mathbf{u}}_1 \to \tilde{\mathbf{U}}_1 \quad \text{and} \quad \tilde{p}_1 \to \tilde{P}_1 \quad \text{as} \quad \tilde{r}\to 0, \tag{3.17}$$

and we seek a solution to (3.14) subject to (3.15) and (3.17). Since this solution is difficult to obtain, it is fortunate that the flow field close to the particle is usually of primary interest and this result can be found if one knows only certain general properties of the outer expansion.

(c) Leading terms of the expansions

In the limit $R \to 0$ the expansions for the velocity field (3.1) and (3.11) reduce to $\mathbf{u} \to \mathbf{u}_0$ and $\tilde{\mathbf{u}} \to \tilde{\mathbf{u}}_0$. For the outer flow we see that the differential equation and the boundary conditions are satisfied by identifying $\tilde{\mathbf{u}}_0$ with the undisturbed flow, $\tilde{\mathbf{u}}_0 = \tilde{y}\mathbf{e}_x$, $\tilde{p}_0 = \tilde{p}_\infty$. The inner solution $\mathbf{u}_0$ is just the Stokes solution to the creeping-motion equations which satisfies the surface boundary condition

$$\mathbf{u}_0 = \mathbf{v}_{0h} + (1/r^3)(\boldsymbol{\Omega}_0 \times \mathbf{r}), \quad p_0 = q_{0h}, \tag{3.18 a, b}$$

where $\mathbf{v}_{0h}$ and q_{0h} are given in appendix A. Constants appearing in $\mathbf{v}_{0h}$ and q_{0h} are found from the requirement that (3.18), when expressed in terms of $\tilde{\mathbf{r}}$, should contain no terms of order, with respect to R, greater than unity. Non-zero constants are $C^0_{0,1} = -\frac{1}{2}$, $B^2_{0,2} = \frac{1}{12}$ and $A^0_{0,0} = p_\infty$. In addition the condition of free suspension (zero net force and torque on the sphere) leads to $\boldsymbol{\Omega}_0 = -\frac{1}{2}\mathbf{e}_z$. Thus

$$\mathbf{u}_0 = \mathbf{e}_x\left[y - \frac{y}{2r^5} - \frac{5x^2y}{2r^5}\left(1-\frac{1}{r^2}\right)\right] + \mathbf{e}_y\left[-\frac{x}{2r^5} - \frac{5}{2}\frac{xy^2}{r^5}\left(1-\frac{1}{r^2}\right)\right] + \mathbf{e}_z\left[-\frac{5}{2}\frac{xyz}{r^5}\left(1-\frac{1}{r^2}\right)\right], \tag{3.19 a}$$

$$p_0 = p_\infty - (5xy/r^5). \tag{3.19 b}$$

(d) Higher terms of the expansions

When (3.19 *a*) is expressed in terms of outer variables, its contribution to $\tilde{\mathbf{u}}$ is

$$\tilde{y}\mathbf{e}_x + R^{\frac{3}{2}}\left[-\frac{5\tilde{x}\tilde{y}\tilde{\mathbf{r}}}{2\tilde{r}^5}\right] + O(R^{\frac{5}{2}}). \tag{3.20}$$

The first term is $\tilde{\mathbf{u}}_0$, while the requirement for matching of inner and outer solutions suggests that

$$F_1(R) = R^{\frac{3}{2}}. \tag{3.21}$$

We next seek the form of $f_1(R)\,\mathbf{u}_1$. One can infer from (A 9 *a*) that $\mathbf{u}_n$ and p_n are linear combinations of terms of the form $\{(x^l y^m z^n/r^k)(\ln r)^j\}$, where j, k, l, m, and n are zero or positive integers. Furthermore, since the transformation factor between inner and outer variables is $R^{\frac{1}{2}}$, $\{f_n(R)\}$ and $\{F_n(R)\}$ must be of the form $\{R^{\frac{1}{2}j}(\ln R)^i\}$, where i and j are integers and $j \geqslant 0$. With this in mind we consider possible inner expansions $R < f_l(R) < 1$; e.g. $R^{\frac{1}{2}}$, $R^{\frac{1}{2}}\ln R$, etc. In all of these cases the constraint on the right-hand side of (3.2 *a*) ensures $\mathbf{g}_l = \mathbf{0}$. Then the general solutions for $\mathbf{u}_l$ and p_l are merely (3.18), with $\mathbf{v}_{0h}$, q_{0h} and $\boldsymbol{\Omega}_0$ replaced by $\mathbf{v}_{lh}$, q_{lh} and $\boldsymbol{\Omega}_l$, respectively. Expressing these results in terms of outer variables and requiring, in view of (3.21), that there be no contribution to $\tilde{\mathbf{u}}$ with a Reynolds number dependence greater than $O(R^{\frac{3}{2}})$, one concludes that

$$\mathbf{v}_{lh} = \mathbf{0}; \quad q_{lh} = 0. \tag{3.22}$$

This result along with the requirement following from the free-suspension condition, viz. $\mathbf{V}_l = \mathbf{\Omega}_l = \mathbf{0}$, ensures that the terms $f_l(R)\,\mathbf{u}_l$ and $f_l(R)\,p_l$ are zero for $R < f_l(R) < 1$. Now we are in a position to consider $f_1(R) = R$. In this case the governing equations for $\mathbf{u}_1$ and p_1 will be (3.2) with $\mathbf{g}_1 = -\mathbf{u}_0 . \nabla\mathbf{u}_0$ and boundary condition (3.3). The general solution is

$$\mathbf{u}_1 = \mathbf{v}_{1h} + (1/r^3)\,(\mathbf{\Omega}_1 \times \mathbf{r}) + \mathbf{u}_{1p}, \quad p_1 = q_{1h} + p_{1p}. \tag{3.23 a, b}$$

The particular solutions $\mathbf{u}_{1,p}$ and $p_{1,p}$ have been derived by Peery (1966) and are available from the authors upon request. From the requirement that no terms in $\mathbf{u}_1$ or p_1 can exist which, when written in outer variables, are larger than $O(R^{\frac{3}{2}})$, one finds

$$\left.\begin{aligned} &A^m_{1,j} = C^m_{1,j} = 0 \quad \text{for all } j, \\ &B^m_{1,j} = 0 \quad \text{for} \quad j \geqslant 2. \end{aligned}\right\} \tag{3.24}$$

Fortunately, and somewhat surprisingly, the first-order outer solution is not required for the complete determination of $\mathbf{u}_1$ and p_1. This is so because $\mathbf{u}_{1p}$ and p_{1p} do not contribute to the force or torque on the particle, and the free-suspension condition requires that $B^m_{1,1} = 0$ and $\mathbf{\Omega}_1 = \mathbf{0}$ (see appendix B). Therefore, $\mathbf{u}_1 = \mathbf{u}_{1p}$, $p_1 = p_{1p}$.

One can readily show that terms of order $R^{\frac{3}{2}}(\ln R)^i$ or $R^{\frac{3}{2}}$ are solutions of the homogeneous Stokes equations. Hence we may write, for the moment, $f_2(R) = R^{\frac{3}{2}}$ and allow the possibility that the arbitrary constants in $\mathbf{v}_{2h}$ and q_{2h} may contain $\ln R$. Application of the matching condition that no terms appear which are larger than $R^{\frac{3}{2}}$ (possibly multiplied by a function of $\ln R$) leads to

$$\left.\begin{aligned} &A^m_{2,j} = 0 \quad \text{all } j, \\ &B^m_{2,j} = 0 \quad (j \geqslant 3), \\ &C^m_{2,j} = 0 \quad (j \geqslant 2). \end{aligned}\right\} \tag{3.25}$$

Then from appendix B,

$$\mathbf{\Omega}_2 = C^{-1}_{2,1}\,\mathbf{e}_x + C^1_{2,1}\,\mathbf{e}_y + C^0_{2,1}\,\mathbf{e}_z, \tag{3.26 a}$$

$$\mathbf{V}_2 = \mathbf{0} = B^{-1}_{2,1}\,\mathbf{e}_x + B^1_{2,1}\,\mathbf{e}_y + B^0_{2,1}\,\mathbf{e}_z, \tag{3.26 b}$$

where values of the non-zero coefficients are to be found from matching inner and outer solutions.

It is convenient to designate contributions of $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ to $\tilde{\mathbf{u}}_1$ by $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, respectively. Then

$$\mathbf{A} = -\frac{5\tilde{x}\tilde{y}}{2\tilde{r}^5}\,\tilde{\mathbf{r}}, \tag{3.27 a}$$

$$\begin{aligned} \mathbf{B} = -\frac{5\tilde{x}^2\tilde{y}^2\tilde{\mathbf{r}}}{24\tilde{r}^5} + \mathbf{e}_x\left[\frac{5\tilde{x}\tilde{y}^2}{24\tilde{r}^3} + \frac{5\tilde{x}^3}{72\tilde{r}^3} - \frac{5\tilde{x}}{24\tilde{r}}\right] \\ + \mathbf{e}_y\left[-\frac{5\tilde{x}^2\tilde{y}}{24\tilde{r}^3} + \frac{5\tilde{y}^3}{72\tilde{r}^3} + \frac{5\tilde{y}}{24\tilde{r}}\right] + \mathbf{e}_z\left[-\frac{5\tilde{z}^3}{72\tilde{r}^3}\right], \end{aligned} \tag{3.27 b}$$

$$\begin{aligned} \mathbf{C} = {} & \mathbf{e}_x[(6B^{-2}_{2,2} - B^0_{2,2})\,\tilde{x} + (6B^2_{2,2} - C^0_{2,1})\,\tilde{y} + (3B^{-1}_{2,2} + C^1_{2,1})\,\tilde{z}] \\ & + \mathbf{e}_y[(6B^2_{2,2} + C^0_{2,1})\,\tilde{x} - (6B^{-2}_{2,2} + B^0_{2,2})\,\tilde{y} + (3B^1_{2,2} - C^{-1}_{2,1})\,\tilde{z}] \\ & + \mathbf{e}_z[(3B^{-1}_{2,2} - C^1_{2,1})\,\tilde{x} + (3B^1_{2,2} + C^{-1}_{2,1})\,\tilde{y} + (2B^0_{2,2})\,\tilde{z}]. \end{aligned} \tag{3.27 c}$$

Following the procedure of Saffman (1965), we express the solution of (3.14a), subject to (3.15), by

$$\tilde{\mathbf{u}}_1 = (\mathbf{H}^{(0)}(\tilde{\mathbf{r}})/\tilde{r}^2) + \mathbf{H}^{(1)}(\tilde{\mathbf{r}}) + \tilde{r}\mathbf{H}^{(2)}(\tilde{\mathbf{r}}) + \tilde{r}^2\mathbf{H}^{(3)}(\tilde{\mathbf{r}}) + \dots, \tag{3.28}$$

where the $\mathbf{H}^{(i)}$ are homogeneous functions of degree zero in $\tilde{x}, \tilde{y}$, and $\tilde{z}$. Matching (3.28) with the inner expansion suggests that $\tilde{\mathbf{u}}_1$ and the inner solutions $\mathbf{u}_0$, $\mathbf{u}_1$, and $\mathbf{u}_2$ are connected by

$$(\mathbf{H}^{(0)}(\tilde{\mathbf{r}})/\tilde{r}^2) = \mathbf{A}, \quad \mathbf{H}^{(1)}(\tilde{\mathbf{r}}) = \mathbf{B}, \quad \tilde{r}\mathbf{H}^{(2)}(\tilde{\mathbf{r}}) = \mathbf{C}. \tag{3.29}$$

One can readily verify that

$$-\tilde{\nabla}\mathscr{P}_0 + \tilde{\nabla}^2\mathbf{A} = \mathbf{0}, \quad \tilde{\nabla}.\mathbf{A} = 0, \tag{3.30a,b}$$

$$-\tilde{\nabla}\mathscr{P}_1 + \tilde{\nabla}^2\mathbf{B} = \tilde{y}\,\partial\mathbf{A}/\partial\tilde{x} + A_y\,\mathbf{e}_x, \quad \tilde{\nabla}.\mathbf{B} = \mathbf{0}, \tag{3.31a,b}$$

where $\mathscr{P}_0$ and $\mathscr{P}_1$ are contributions of p_0 and p_1, respectively, to $\tilde{p}_1$.

Combining (3.14), (3.30), and (3.31), we obtain

$$\tilde{\nabla}(\tilde{p}_1 - \mathscr{P}_0 - \mathscr{P}_1) + \left[\left(\tilde{y}\frac{\partial}{\partial\tilde{x}} - \nabla^2\right)\mathbf{I} + \mathbf{e}_x\mathbf{e}_y\right].(\tilde{\mathbf{u}}_1 - \mathbf{A} - \mathbf{B}) = -\tilde{y}\frac{\partial\mathbf{B}}{\partial\tilde{x}} - B_y\,\mathbf{e}_x, \tag{3.32a}$$

$$\tilde{\nabla}.(\tilde{\mathbf{u}}_1 - \mathbf{A} - \mathbf{B}) = 0. \tag{3.32b}$$

The Fourier transformation of (3.32) is

$$i\mathbf{k}\Pi + k^2\mathbf{\Gamma} - k_x\frac{\partial\mathbf{\Gamma}}{\partial k_y} + \Gamma_y\,\mathbf{e}_x = k_x\frac{\partial\mathbf{\Gamma}^{(1)}}{\partial k_y} - \Gamma_y^{(1)}\,\mathbf{e}_x, \tag{3.33a}$$

$$\mathbf{k}.\mathbf{\Gamma} = 0, \tag{3.33b}$$

where
$$\mathbf{\Gamma}(\mathbf{k}) = \frac{1}{8\pi^3}\int(\tilde{\mathbf{u}}_1 - \mathbf{A} - \mathbf{B})\,e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,d\tilde{\mathbf{r}},$$

$$\Pi(\mathbf{k}) = \frac{1}{8\pi^3}\int(\tilde{p}_1 - \mathscr{P}_0 - \mathscr{P}_1)\,e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,d\tilde{\mathbf{r}},$$

$$\mathbf{\Gamma}^{(1)}(\mathbf{k}) = \frac{1}{8\pi^3}\int\mathbf{B}\,e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,d\tilde{\mathbf{r}} = \frac{5i}{3\pi^2}\mathbf{k}\left(\frac{k_x^2k_y^2}{k^8}\right) - \frac{5i}{3\pi^2}\mathbf{e}_y\left(\frac{k_yk_x^2}{k^6}\right).$$

From (3.33) one readily obtains a set of equations for the components of $\mathbf{\Gamma}(\mathbf{k})$,

$$k_x\frac{\partial\Gamma_x}{\partial k_y} - k^2\Gamma_x = \left(1 - \frac{2k_x^2}{k^2}\right)\Gamma_y + \frac{5i}{3\pi^2}\left[\frac{6k_x^2k_y^2}{k^4} + \frac{k_y^2}{k^2} - 1\right]\frac{k_x^2k_y}{k^6} = S_x(\mathbf{k}), \tag{3.34a}$$

$$k_x\frac{\partial\Gamma_y}{\partial k_y} + \left(\frac{2k_xk_y}{k^2} - k^2\right)\Gamma_y = \frac{5i}{3\pi^2}\frac{k_x^3(k_x^2+k_z^2)}{k^8}\left[1 - \frac{6k_y^2}{k^2}\right] = S_y(\mathbf{k}), \tag{3.34b}$$

and
$$k_x\frac{\partial\Gamma_z}{\partial k_y} - k^2\Gamma_z = -\frac{2k_xk_z}{k^2}\Gamma_y + \frac{10i}{\pi^2}\left[\frac{k_x^3k_y^3k_z}{k^{10}}\right] = S_z(\mathbf{k}), \tag{3.34c}$$

the solutions to which are given by

$$\Gamma_x = -\int_0^\infty S_x(k_x, k_x\xi + k_y, k_z)\,e^{-\beta}\,d\xi, \tag{3.35a}$$

$$\Gamma_y = -\frac{1}{k^2}\int_0^\infty (k^2 + 2k_x k_y \xi + k_x^2 \xi^2) S_y(k_x, k_x\xi + k_y, k_z)\, e^{-\beta}\, d\xi, \qquad (3.35\,b)$$

$$\Gamma_z = -\int_0^\infty S_z(k_x, k_x\xi + k_y, k_z)\, e^{-\beta}\, d\xi, \qquad (3.35\,c)$$

where $$\beta = \xi(k^2 + k_x k_y \xi + \tfrac{1}{3}k_x^2 \xi^2).$$

We now consider the possibility that terms of order $R^{\frac{3}{2}} \ln R$ exist in the inner expansion. This is readily checked by studying the behaviour of

$$\nabla(\tilde{\mathbf{u}}_1 - \mathbf{A} - \mathbf{B}) \to \nabla\mathbf{C} \text{ as } \tilde{r} \to 0.$$

If $(\nabla\mathbf{C})_{\tilde{r}\to 0}$ is bounded, then we conclude that no logarithmic term is present, and the undetermined coefficients in $\mathbf{u}_2$ are independent of Reynolds number. It will be shown below that the equivalent of $(\nabla\mathbf{C})_{\tilde{r}\to 0}$, viz.

$$\int_{-\infty}^{\infty} i\mathbf{k}\boldsymbol{\Gamma}\, d\mathbf{k},$$

is indeed bounded. Thus we conclude that the Reynolds number dependence of $f_2(R)\,\mathbf{u}_2$ is $R^{\frac{3}{2}}$.

Our next step is to evaluate the undetermined constants in $\mathbf{u}_2$ by employing an adaptation of the Fourier transformation technique of Childress (1964) and Saffman (1965).

From (3.28) one can write

$$[\tilde{\nabla}(\tilde{\mathbf{u}}_1 - \mathbf{A} - \mathbf{B})]_{\tilde{\mathbf{r}}=0} = (\tilde{\nabla}\mathbf{C}) = \int_{-\infty}^{\infty} i\mathbf{k}\boldsymbol{\Gamma}\, d\mathbf{k}. \qquad (3.36)\dagger$$

From (3.35) it is apparent that

$$\int_{-\infty}^{\infty} ik_z \Gamma_x\, d\mathbf{k} = \int_{-\infty}^{\infty} ik_z \Gamma_y\, d\mathbf{k} = \int_{-\infty}^{\infty} ik_x \Gamma_z\, d\mathbf{k} = \int_{-\infty}^{\infty} ik_y \Gamma_z\, d\mathbf{k} = 0,$$

and it then follows from (3.27 c) that $B_{2,2}^{-1} = B_{2,2}^{1} = C_{2,1}^{1} = C_{2,1}^{-1} = 0$. From (3.27 c) one can now write

$$\mathbf{C} = \mathbf{e}_x[(6B_{2,2}^{-2} - B_{2,2}^{0})\tilde{x} + (6B_{2,2}^{2} - C_{2,1}^{0})\tilde{y}] + \mathbf{e}_y[(6B_{2,2}^{2} + C_{2,1}^{0})\tilde{x} - (6B_{2,2}^{-2} + B_{2,2}^{0})\tilde{y}] + \mathbf{e}_z[2B_{2,2}^{0}\tilde{z}],$$

so that $$\int_{-\infty}^{\infty} i\mathbf{k}\boldsymbol{\Gamma}\, d\mathbf{k} = \begin{bmatrix} 6B_{2,2}^{-2} - B_{2,2}^{0} & 6B_{2,2}^{2} + C_{2,1}^{0} & 0 \\ 6B_{2,2}^{2} - C_{2,1}^{0} & -(6B_{2,2}^{-2} + B_{2,2}^{0}) & 0 \\ 0 & 0 & 2B_{2,2}^{0} \end{bmatrix}. \qquad (3.37)$$

The unknown coefficients can be found from numerical integration of four independent components of the left-hand side of (3.37). The integrals chosen were

$$\int_{-\infty}^{\infty} ik_x \Gamma_y\, d\mathbf{k}, \quad \int_{-\infty}^{\infty} ik_y \Gamma_y\, d\mathbf{k}, \quad \int_{-\infty}^{\infty} ik_z \Gamma_z\, d\mathbf{k}, \quad \text{and} \quad \int_{-\infty}^{\infty} ik_y \Gamma_x\, d\mathbf{k}.$$

† Whenever a quantity having physical significance is related to a complex expression, we of course refer only to the real part.

The first two can be reduced to double definite integrals, while the last two involve both double and triple integrals. Details of the reduction are available from the authors upon request. One obtains

$$B_{2,2}^{-2} = 0{\cdot}00479, \quad B_{2,2}^{0} = 0{\cdot}00724, \quad B_{2,2}^{2} = 0{\cdot}0448,$$
$$C_{2,1}^{0} = 0{\cdot}1538,$$

along with the requirement, which follows from the condition of zero translational velocity, that $B_{2,1}^{-1} = B_{2,1}^{0} = B_{2,1}^{1} = 0$.

Employing (A 5) we finally have

$$\begin{aligned}
\mathbf{u}_2 = \mathbf{e}_x &\left\{x\left[6B_{2,2}^{-2}\left(1-\frac{1}{r^5}\right)-B_{2,2}^{0}\left(1+\frac{5}{r^3}-\frac{6}{r^5}\right)\right]+y\left[6B_{2,2}^{2}\left(1-\frac{1}{r^5}\right)-C_{2,1}^{0}\right]\right\} \\
&+\mathbf{e}_y x\left[6B_{2,2}^{2}\left(1-\frac{1}{r^5}\right)+C_{2,1}^{0}\right]+y\left[6B_{2,2}^{-2}\left(\frac{1}{r^5}-1\right)-B_{2,2}^{0}\left(1+\frac{5}{r^3}-\frac{6}{r^5}\right)\right]\Big\} \\
&+\mathbf{e}_z\left[2-\frac{5}{r^3}+\frac{3}{r^5}\right]B_{2,2}^{0}z+\frac{15\mathbf{r}}{2r^5}\left(\frac{1}{r^2}-1\right) \\
&\times\left[(2B_{2,2}^{-2}-B_{2,2}^{0})\,x^2-(2B_{2,2}^{-2}+B_{2,2}^{0})\,y^2+4B_{2,2}^{2}\,xy\right]
\end{aligned} \tag{3.38a}$$

$$p_2 = -\frac{5}{r^5}\left[6B_{2,2}^{-2}(x^2-y^2)+B_{2,2}^{0}(3z^2-r^2)+12B_{2,2}^{2}\,xy\right]. \tag{3.38b}$$

4. Streamlines around a single sphere

The velocity field in the neighbourhood of a sphere can now be evaluated to $O(R^{\frac{3}{2}})$ from

$$\mathbf{u} = \mathbf{u}_0 + R\mathbf{u}_1 + R^{\frac{3}{2}}\mathbf{u}_2 \tag{4.1}$$

with surface boundary conditions

$$\mathbf{V} = \mathbf{0}, \quad \mathbf{\Omega} = -(\tfrac{1}{2} - 0{\cdot}1538R^{\frac{3}{2}})\,\mathbf{e}_z. \tag{4.2a, b}$$

The solution of (4.1) which meets boundary conditions (4.2) is of course valid only in a region near the sphere, the extent of this region increasing with decreasing R. Streamlines may be found from solution of the defining equations

$$dx/u_x = dy/u_y = dz/u_z. \tag{4.3}$$

For illustrative purposes several streamlines have been computed which lie in the plane $z = 0$. Since an explicit solution to the outer expansion is not known beyond the first term, the streamlines computed from (4.1) and (4.3) are only valid close to the sphere. The solution was obtained by numerical integration, using a Runge–Kutta technique, of

$$(dy/dx)_{z=0} = (u_y/u_x)_{z=0}.$$

Some of the results for $R = 0{\cdot}5$ are shown in figures 1 and 2. A comparison between streamlines of the creeping motion solution and the first-order approximation (to $O(R)$) for $\mathbf{V} = 0$ and $\mathbf{\Omega} = -\frac{1}{2}\mathbf{e}_z$, and between the creeping-motion solution and second-order approximation (to $O(R^{\frac{3}{2}})$) with surface conditions (4.2) is shown in figure 1 and figure 2, respectively. For both cases the two families of streamlines are matched on the y axis. Because of the antisymmetric nature of the flow, only the region $y \geqslant 0$ is shown. It is seen that the fore–aft symmetry of the creeping-flow solution is destroyed when the effect of Reynolds number is considered. The restricted region of applicability of the inner expansion is also clearly shown. Though the creeping-motion solution is bounded in the whole domain, one notes from figure 2 that the second-order approximation to the inner expansion severely distorts the streamlines at sufficiently large r. Fortunately, one is usually most interested in the contribution of inertia to the flow field near the sphere, where the second-order correction is valid.

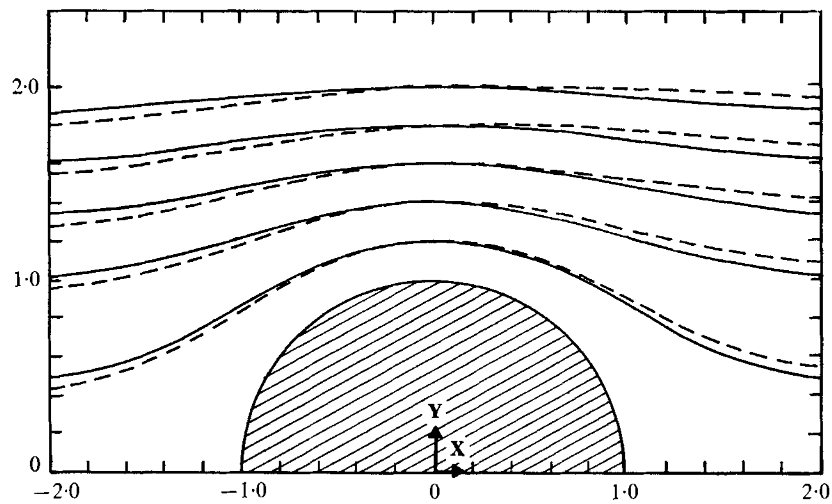

FIGURE 1. Streamlines in the plane $z = 0$, $R = 0{\cdot}50$, $\mathbf{\Omega} = -\frac{1}{2}\mathbf{e}_z$. Solid lines represent creeping-motion solution; dashed lines represent first-order approximation ($O(R)$).

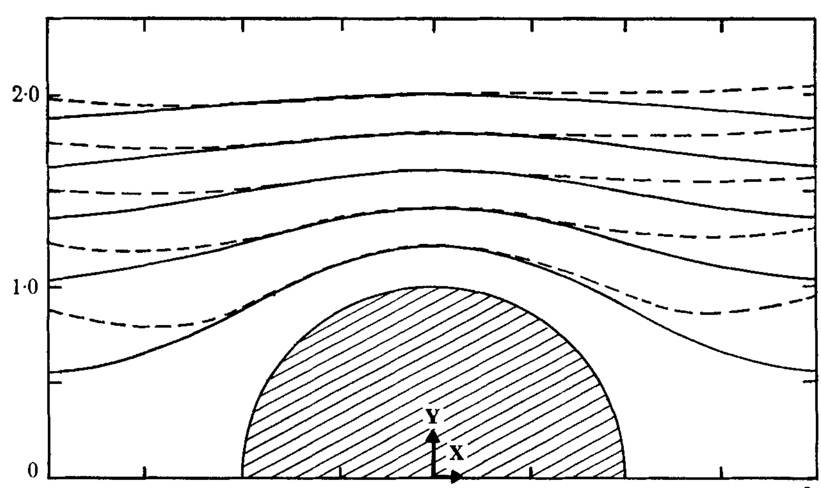

FIGURE 2. Streamlines in the plane $z = 0$. $R = 0{\cdot}50$, $\mathbf{\Omega} = -(\frac{1}{2} - 0{\cdot}1538R^{\frac{3}{2}})\mathbf{e}_z$. Solid lines represent creeping-motion solution; dashed lines represent second-order approximation ($O(R^{\frac{3}{2}})$).

5. Suspension rheology

Attempts to relate the microscopic flow field around a single particle to the macroscopic behaviour of an assembly of particles in a continuous fluid phase have been numerous and varied, and begin with the classic work of Einstein (1906, 1911). Batchelor (1970) has provided a new assessment of the means by which one proceeds from a knowledge of microscopic flow behaviour to an expression of bulk stress in a suspension. In particular, the contribution which a momentum flux term can make to an expression for the bulk stress has been shown.

Following earlier workers we consider a dilute homogeneous suspension of neutrally-buoyant particles which is subjected to steady shear. A connexion is then made between the stress field around a single particle and the average stress and average strain rate of the suspension. The purpose of the present computation is to show the effect of a non-zero Reynolds number on the rheological behaviour of the suspension. For the dilute (non-interacting) suspension of rigid spheres which we are considering, the dimensionless stress tensor $\langle \mathbf{t} \rangle$ of the suspension in plane Couette flow may be represented in terms of the stress on a single sphere. The expression for bulk stress given by Batchelor (1970) reduces, in dimensionless form, to

$$\langle \mathbf{t} \rangle + T\mathbf{I} = (\mathbf{e}_x \mathbf{e}_y + \mathbf{e}_y \mathbf{e}_x) + \frac{3\phi}{4\pi}\left\{ \int_0^{2\pi} \int_0^{\pi} \mathbf{t}^a . \mathbf{e}_r \mathbf{e}_r \sin\theta \, d\theta \, d\varphi \right.$$
$$\left. - R\int_{V_s} \mathbf{a r} \, dV - R\int_{V_s + V_{fl}} (\mathbf{u} - \mathbf{e}_x y)(\mathbf{u} - \mathbf{e}_x y) \, dV \right\}, \quad (5.1)$$

where ϕ is the volume fraction of the spheres and $\mathbf{t}^a$ is the dimensionless stress tensor on the surface of a single sphere. In addition to the terms familiar from earlier treatments of bulk stress (see, for example, Landau & Lifshitz 1959), two additional contributions arise which are $O(R)$. The first is a contribution to bulk stress from local acceleration $\mathbf{a}$ within the spheres. Integration is over the volume V_s of one sphere. The second, representing a momentum flux contribution, is an integral taken over a single sphere and the surrounding fluid field. Since the fluid is incompressible, the stress can only be specified to within an arbitrary isotropic stress $T\mathbf{I}$.

On the surface of the sphere we have

$$\mathbf{t} . \mathbf{e}_r = \mathbf{t}_r = -\mathbf{e}_r p + \left(\frac{\partial \mathbf{u}}{\partial r} - \frac{\mathbf{u}}{r} \right) + \frac{1}{r} \nabla(\mathbf{r} . \mathbf{u}), \quad (5.2)$$

which can be combined with

$$\mathbf{u} = \mathbf{u}_0 + R\mathbf{u}_1 + R^{\frac{3}{2}}\mathbf{u}_2 + o(R^{\frac{3}{2}}), \quad (5.3a)$$

$$p = p_0 + Rp_1 + R^{\frac{3}{2}}p_2 + o(R^{\frac{3}{2}}), \quad (5.3b)$$

to give

$$\mathbf{t}_r^a = \mathbf{t}_{r0}^a + R\mathbf{t}_{r1}^a + R^{\frac{3}{2}}\mathbf{t}_{r2}^a + o(R^{\frac{3}{2}}). \quad (5.4)$$

Combining (5.1) to (5.4) and evaluating the integrals (appendix B) one finally obtains the rectangular Cartesian components of $\langle \mathbf{t} \rangle$ to $O(R^{\frac{3}{2}})$. Expressed in dimensional form we have, to $O(\phi)$,

$$[\langle \mathbf{t}' \rangle + T'\mathbf{I}] = \mu G \begin{bmatrix} R\phi[-\frac{169}{420}-\frac{97}{420} + (30B^{-2}_{2,2}-5B^{0}_{2,2})R^{\frac{1}{2}}] & 1+\frac{5}{2}\phi+30B^{2}_{2,2}R^{\frac{3}{2}}\phi & 0 \\ 1+\frac{5}{2}\phi+30B^{2}_{2,2}R^{\frac{3}{2}}\phi & R\phi[-\frac{169}{420}+\frac{463}{420} - (30B^{-2}_{2,2}+5B^{0}_{2,2})R^{\frac{1}{2}}] & 0 \\ 0 & 0 & R\phi[-\frac{40}{420}+\frac{54}{420} + 10B^{0}_{2,2}R^{\frac{1}{2}}] \end{bmatrix} \tag{5.5}$$

where G is now the macroscopic shear rate (Batchelor 1970). Several interesting consequences of inertial effects are revealed in the computation of (5.5) from (5.1). One finds that the acceleration term in (5.1) is exactly cancelled by the momentum flux integration over the sphere volume V_s. The remaining contribution from momentum flux appears only in the diagonal components of the stress tensor and is represented by the first fraction appearing in each of the diagonal components of (5.5). Since we are only interested in terms of $O(R^{\frac{3}{2}})$ or larger, the momentum flux integral can be evaluated using $\mathbf{u}_0$ from (3.19a), and the lack of an outer solution beyond that of the uniform shear field is of no consequence. Although an inertial correction to $O(R)$ appears in the normal stress components, we note that the Einstein correction to the shear viscosity is only altered to $O(R^{\frac{3}{2}})$. Inserting the numerical values of $B^{i}_{2,2}$ we obtain the suspension viscosity

$$\mu_s = \mu[1+\phi(\tfrac{5}{2}+1\cdot34R^{\frac{3}{2}})] \tag{5.6a}$$

and dimensional normal stresses t'_{ij}, expressed as stress differences,

$$t'_{xx}-t'_{zz} = \mu G\phi R[-\tfrac{2}{3}+0\cdot035R^{\frac{1}{2}}], \tag{5.6b}$$

$$t'_{yy}-t'_{zz} = \mu G\phi R[\tfrac{2}{3}-0\cdot252R^{\frac{1}{2}}]. \tag{5.6c}$$

Generalization of (5.6) to an arbitrary homogeneous shear field $\partial u_i/\partial x_j$ appears to require further computation. Professor Acrivos has pointed out to the authors that the $O(R)$ correction to the Einstein expression must be quadratic in $\partial u_i/\partial x_j$ (used here to denote the bulk velocity gradient) and, consequently, one expects the correction, to within some arbitrary isotropic part, to have the form

$$\alpha_1\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_j}+\frac{\partial u_j}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right)+\alpha_2\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)+\alpha_3\left(\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right).$$

In the case treated here ($\partial u_i/\partial x_j = 0$ except for $i = 1$, $j = 2$) the coefficient of α_1 is zero.

Unfortunately, there are no data available to which these predictions for rheological behaviour of dilute suspensions can be compared. The importance of the calculation, which is of course valid only at low Reynolds numbers, is primarily in its qualitative features. For example, we find that the effect of inertia at sufficiently low R is to cause the suspension viscosity to *increase* with increasing shear rate. Furthermore, it is apparent from (5.6) that the so-called 'secondary' normal stress difference ($t'_{yy}-t'_{zz}$) is positive and has a magnitude comparable to the negative 'primary' normal stress difference $(t'_{xx}-t'_{zz})$. That these results are quite different from the observed behaviour of most dilute polymer solutions, which typically display both a shear dependent viscosity and unequal normal stress components, is not surprising. To effect correspondence between rheological behaviour of a suspension and a polymer solution one must evidently, at the very least, account for deformation of the discontinuous phase. An initial attempt to account for deformation has been made by Schowalter *et al.* (1968).

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Appendix A. Solution of the creeping-motion equations

Consider first the homogeneous equations

$$-\nabla p_{nh}+\nabla^2\mathbf{u}_{nh}=\mathbf{0},\quad \nabla.\mathbf{u}_{nh}=\mathbf{0} \tag{A 1a,b}$$

with boundary conditions

$$\mathbf{u}_{nh}=\mathbf{V}_n+\mathbf{\Omega}_n\times\mathbf{r}\quad\text{at}\quad r=1. \tag{A 2†}$$

The general solution to (A 1) can be expressed in spherical co-ordinates in the form presented by Lamb (1945)

$$\mathbf{u}_{nh}=\sum_{j=-\infty}^{\infty}\left\{\nabla\times(\mathbf{r}\chi_{n,j})+\nabla\Phi_{n,j}+\frac{(j+3)}{2(j+1)(2j+3)}r^2\nabla Q_{n,j}-\frac{j}{(j+1)(2j+3)}\mathbf{r}Q_{n,j}\right\}, \tag{A 3a}$$

$$p_{nh}=\sum_{j=-\infty}^{\infty}Q_{n,j}, \tag{A 3b}$$

where $\chi_{n,j}$, $\Phi_{n,j}$, and $Q_{n,j}$ are each solid spherical harmonics.

Boundary conditions may be incorporated into the solution (A 3) by employing the techniques of Happel & Brenner (1965, p. 62 *et seq.*). One eventually obtains

$$\chi^a_{n,-(j+1)}=-\chi^a_{n,j}+\left(\frac{\mathbf{r}}{r}.\mathbf{\Omega}_n\right)\delta_{j1}, \tag{A 4a}$$

$$\Phi^a_{n,-(j+1)}=-\frac{j(2j+1)}{4(j+1)(2j+3)}Q^a_{n,j}-\frac{j(2j-1)}{2(j+1)}\Phi^a_{n,j}+\frac{1}{4}\left(\frac{\mathbf{r}}{r}.\mathbf{V}_n\right)\delta_{j1}, \tag{A 4b}$$

$$Q^a_{n,-(j+1)}=-\frac{j(2j-1)}{2(j+1)}Q^a_{n,j}-\frac{j(2j+1)(2j-1)}{j+1}\Phi^a_{n,j}+\frac{3}{2}\left(\frac{\mathbf{r}}{r}.\mathbf{V}_n\right)\delta_{j1}. \tag{A 4c}$$

for $j\geqslant 1$, where δ_{j1} is the Kronecker delta and the $Q^a_{n,j}$, are surface spherical harmonics defined by $Q_{n,j}=r^jQ^a_{n,j}$. $\chi^a_{n,j}$ and $\Phi^a_{n,j}$ are similarly defined.

† Though in the present problem the velocity of translation $\mathbf{V}=\mathbf{0}$, it is convenient to retain contributions $\mathbf{V}_n$ to $\mathbf{V}$ in this general development.

From (A 3) and (A 4),

$$\mathbf{u}_{nh} = \mathbf{v}_{nh} + \mathbf{V}_n\left(\frac{3}{4r}+\frac{1}{4r^3}\right) + \frac{3}{4}\frac{\mathbf{r}}{r^3}\left(1-\frac{1}{r^2}\right)(\mathbf{V}_n\cdot\mathbf{r}) + \frac{1}{r^3}\boldsymbol{\Omega}_n\times\mathbf{r}, \qquad \text{(A 5}a)$$

$$p_{nh} = q_{nh} + \frac{3}{2r^3}(\mathbf{V}_n\cdot\mathbf{r}), \qquad \text{(A 5}b)$$

where

$$\mathbf{v}_{nh} = \sum_{j=1}^{\infty}\left\{\nabla\left[\chi^a_{n,j}\left(r^j-\frac{1}{r^{j+1}}\right)\right]\times\mathbf{r} + \nabla\left[\Phi^a_{n,j}\left(r^j+\frac{(2j+1)(j-2)}{2(j+1)r^{j-1}}-\frac{j(2j-1)}{2(j+1)r^{j+1}}\right)\right.\right.$$
$$\left.+Q^a_{n,j}\left(\frac{(j+3)r^{j+2}}{2(j+1)(2j+3)}+\frac{j-2}{4(j+1)r^{j-1}}-\frac{j(2j+1)}{4(2j+3)(j+1)r^{j+1}}\right)\right]$$
$$\left.-\frac{(2j+1)(2j-1)}{(j+1)r^{j+1}}\mathbf{r}\Phi^a_{n,j} - \mathbf{r}Q^a_{n,j}\left(\frac{r^j}{j+1}+\frac{2j-1}{2(j+1)r^{j+1}}\right)\right\}, \qquad \text{(A 5}c)$$

$$q_{nh} = \sum_{j=1}^{\infty}\left\{\left[r^j-\frac{j(2j-1)}{2(j+1)r^{j+1}}\right]Q^a_{n,j} - \frac{j(2j+1)(2j-1)}{(j+1)r^{j+1}}\Phi^a_{n,j}\right\}. \qquad \text{(A 5}d)$$

It is convenient to make the substitutions

$$\{Q^a_{n,j}, \Phi^a_{n,j}, \chi^a_{n,j}\} = \sum_{m=-j}^{j}\{A^m_{n,j}, B^m_{n,j}, C^m_{n,j}\}\, Y^m_j(\theta,\phi), \qquad \text{(A 6)}$$

where
$$Y^m_j(\theta,\phi) = \begin{cases} P^{|m|}_j(\cos\theta)\cos(m\phi) & (m\leqslant 0) \\ P^{|m|}_j(\cos\theta)\sin(m\phi) & (m\geqslant 1). \end{cases}$$

$P^{|m|}_j(\cos\theta)$ is the associated Legendre polynomial of the first kind of order j and rank $|m|$.

We also desire a solution to the *inhomogeneous* Stokes equations

$$-\nabla p_{np} + \nabla^2\mathbf{u}_{np} = -\mathbf{g}_n(\mathbf{r}), \quad \nabla\cdot\mathbf{u}_{np} = 0, \qquad \text{(A 7}a, b)$$

which satisfies the boundary condition

$$\mathbf{u}_{np} = \mathbf{0} \quad \text{at} \quad r = 1. \qquad \text{(A 8)}$$

By using dyadic Green's functions (Morse & Feshbach 1953, p. 1769 *et seq.*) one can show that

$$\mathbf{u}_{np} = \mathbf{u}^0_{np} + \int_{\mathbf{k}}\frac{1}{k^2}e^{i\mathbf{k}\cdot\mathbf{r}}\left[\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right]\cdot\mathscr{g}_n(\mathbf{k})\,d\mathbf{k}, \qquad \text{(A 9}a)$$

$$p_{np} = p^0_{np} - \int_{\mathbf{k}}\frac{i}{k^2}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\cdot\mathscr{g}_n(\mathbf{k})\,d\mathbf{k} \qquad \text{(A 9}b)$$

where $\mathscr{g}_n(\mathbf{k})$ is the Fourier transform of $\mathbf{g}_n(\mathbf{r})$

$$\mathscr{g}_n(\mathbf{k}) = \frac{1}{8\pi^3}\int_{\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{g}_n(\mathbf{r})\,d\mathbf{r}.$$

The quantities $\mathbf{u}^0_{np}$ and p^0_{np} are given by expressions of the form (A 3). Coefficients of the harmonics are chosen so that the boundary condition (A 8) is satisfied.

Appendix B. Consequences of the free-suspension boundary condition

Following the division of velocity and pressure into homogeneous and particular parts (3.4), we can designate the contributions of these quantities to the stress vector (5.2) on the surface of the sphere by

$$\mathbf{t}_r = \Sigma f_n(R)\,\mathbf{t}_{rn} = \Sigma f_n(R)\,(\mathbf{t}_{rnh}+\mathbf{t}_{rnp}). \tag{B 1}$$

Then, from the condition of free suspension (3.8) we have

$$\mathbf{F}_n = \mathbf{F}_{nh}+\mathbf{F}_{np} = \mathbf{0} = \int_S \mathbf{t}_{rnh}\,dS+\int_S \mathbf{t}_{rnp}\,dS, \tag{B 2a}$$

$$\mathbf{T}_n = \mathbf{T}_{nh}+\mathbf{T}_{np} = \mathbf{0} = \int_S \mathbf{r}\times\mathbf{t}_{rnh}\,dS+\int_S \mathbf{r}\times\mathbf{t}_{rnp}\,dS. \tag{B 2b}$$

For the problem under consideration here we have established (§ 3) that the inner solution has the following homogeneous and particular parts:

$$\{\mathbf{u}_0, p_0\} = \{\mathbf{u}_{0h}, p_{0h}\},$$

$$\{\mathbf{u}_1, p_1\} = \{\mathbf{u}_{1p}, p_{1p}\},$$

$$\{\mathbf{u}_2, p_2\} = \{\mathbf{u}_{2h}, p_{2h}\}.$$

One can verify, by direct substitution of $\mathbf{u}_{1p}$ and p_{1p} into (5.2) and integration of the particular portions of (B 2), that $\mathbf{F}_{1p} = \mathbf{T}_{1p} = \mathbf{0}$. Thus, to the order of approximation employed here, we need only be concerned with the homogeneous contributions to $\mathbf{F}$ and $\mathbf{T}$. From Happel & Brenner (1965) we can write for a spherical surface

$$\mathbf{t}_{rnh} = \mathbf{t}_{nh}\cdot\mathbf{e}_r = \frac{1}{r}\sum_{j=-\infty}^{\infty}\Bigg[(j-1)\,\nabla\times(\mathbf{r}\chi_{n,j})+2(j-1)\,\nabla\Phi_{n,j} - \frac{(2j^2+4j+3)}{(j+1)(2j+3)}\,\mathbf{r}Q_{n,j}+\frac{j(j+2)}{(j+1)(2j+3)}\,r^2\nabla Q_{n,j}\Bigg], \tag{B 3}$$

from which it readily follows that

$$\mathbf{F}_{nh} = \mathbf{0} = \int \mathbf{t}_{rnh}\,dS = -4\pi\nabla(r^3Q_{n,-2}), \tag{B 4a}$$

$$\mathbf{T}_{nh} = \mathbf{0} = \int \mathbf{r}\times\mathbf{t}_{rnp}\,dS = -8\pi\nabla(r^3\chi_{n,-2}). \tag{B 4b}$$

Combining (B 4) with (A 4a) and (A 4c), we have

$$\mathbf{\Omega}_n = C^{-1}_{n,1}\,\mathbf{e}_x+C^{1}_{n,1}\,\mathbf{e}_y+C^{0}_{n,1}\,\mathbf{e}_z \tag{B 5}$$

and $$\mathbf{V}_n = \mathbf{0} = (\tfrac{1}{6}A^{-1}_{n,1}+B^{-1}_{n,1})\,\mathbf{e}_x+(\tfrac{1}{6}A^{1}_{n,1}+B^{1}_{n,1})\,\mathbf{e}_y+(\tfrac{1}{6}A^{0}_{n,1}+B^{0}_{n,1})\,\mathbf{e}_z. \tag{B 6}$$

Recall that in § 3 all $A^i_{n,1}$ were shown to be zero for $n = 0, 1, 2$.

We next consider evaluation of the first integral in (5.1). The contribution from $\mathbf{t}_{0h}$ leads to the well-known Einstein correction. The contribution from $\mathbf{t}_{1p}$ was found by direct calculation using $\mathbf{u}_{1p}$, p_{1p} and (5.2). Evaluation of the effect of $\mathbf{t}_{2h}$ is facilitated through rearrangement of (B 3), which can be written to apply at $r = 1$ as

$$\mathbf{t}^a_{rnh} = \sum_{j=1}^{\infty} \left\{ (j-1)\,[\nabla \times (\mathbf{r}\chi_{n,j})]^a - (j+2)\,[\nabla \times (\mathbf{r}\chi_{n,-j-1})]^a \right.$$
$$\left. + \frac{(2j+1)\,(2j-1)}{(j+1)} [\nabla \Phi_{n,j}]^a + \frac{2j+1}{2(j+1)} [\nabla Q_{n,j}]^a - \frac{2j+1}{j+1} \mathbf{e}_r Q^a_{n,j} \right\}. \quad \text{(B 7)}$$

One can readily show that for an arbitrary solid spherical harmonic H_j of order j,

$$\int_S \mathbf{e}_r [\nabla H_j]^a \, dS = \mathbf{0} \quad \text{for} \quad j \neq 2, \qquad \text{(B 8}a\text{)}$$

$$\int_S \mathbf{e}_r [\nabla \times (\mathbf{r} H_j)]^a \, dS = \mathbf{0} \quad \text{for} \quad j \neq 1 \quad \text{or} \quad -2. \qquad \text{(B 8}b\text{)}$$

Using (B 8) along with the knowledge (§ 3) that $Q^a_{2,j} = 0$, one can write

$$\int_0^{2\pi} \int_0^{\pi} \mathbf{t}^a_{r2h}\, \mathbf{e}_r \sin\theta \, d\theta \, d\phi = 5 \int_S \mathbf{e}_r [\nabla \Phi_{2,2}]^a \, dS. \qquad \text{(B 9)}$$

Carrying out the integration one obtains, in rectangular Cartesian components,

$$\pi \begin{bmatrix} 40B^{-2}_{2,2} - \frac{20}{3}B^0_{2,2} & 40B^0_{2,2} & 0 \\ 40B^0_{2,2} & -(40B^{-2}_{2,2} + \frac{20}{3}B^0_{2,2}) & 0 \\ 0 & 0 & \frac{40}{3}B^0_{2,2} \end{bmatrix}.$$

Combination of contributions to $\langle \mathbf{t} \rangle$ from different orders in R leads to (5.5).

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